supersonic flow the ballistic coefficients of a spherical drop and a cylindrical element are nearly equal.

We conclude on the basis of this discussion that the fundamental similarity of liquid jet penetration to drop breakup is established quantitatively. Other similarities probably can be found. For example, if the shattering results from the consumption of liquid mass by the unstable growth of surface waves,[1] then the breakup time of a liquid element should be proportional to the volume-to-wetted-area ratio of the element. This ratio has the proportions of 2 to 3 for a sphere with respect to a cylindrical element, in good agreement with the ratio of respective breakup times, which are about 4 to 6, respectively. As another example, the jet penetration regimes defined in Ref. 2 may correspond to modes of drop breakup identified in Ref. 1. Apparent dissimilarities also exist, however. An important example is the role played by surface tension. Both theory and experiment indicate that drop breakup has a weak dependence on surface tension,[1,3] while in Ref. 5 it is concluded on the basis of experimental data that surface tension has no influence on jet penetration.

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Multiburst Cloud Rise

Alan W. Zimmerman*
TRW Defense and Space Systems Group, Redondo Beach, Calif.

Nomenclature

a = cloud aspect ratio $(2b/\Delta h)$
b = cloud horizontal radius
F = $(2\pi b^3/a)g[(\rho_0-\rho)/\rho_I]$
G = $-(g/\rho_I)(\mathrm{d}\rho_0/\mathrm{d}x)$
g = acceleration due to gravity
H = stabilization altitude of cloud
Δh = cloud thickness
N = number of bursts
R = nondimensional cloud radius (see Table 1)
s = burst separation
t = time
t_I = nondimensional time (see Table 1)
U = nondimensional vertical velocity (see Table 1)
u = mean vertical velocity of cloud
W = nuclear weapon yield
x = altitude
X = nondimensional altitude (see Table 1)
α = entrainment constant
β = $b_0^2/N(s^2/4)$
Δ = nondimensional cloud buoyancy (see Table 1)
ρ = average density inside cloud
ρ_0 = ambient density at cloud level
ρ_I = ambient density at source level

Subscripts

ES = extended source
MB = multiburst
0 = initial value, except ρ_0
PS = point source
SB = single burst

Received Nov. 10, 1977.
Index category: Hydrodynamics.
*Member of Technical Staff. Member AIAA.

Introduction

THE dimensions and stabilization altitude of the cloud produced by a near-surface nuclear burst are of interest to the weapons system designer. The gross motion of the cloud, except at very early times, is like that of a buoyant thermal, and solutions[1-3] are available for a single burst. Equivalent models are not available for the case where two or more rising clouds interact. It is the purpose of this Note to present an extension of the instantaneous point source solution of Ref. 1 to a limiting case of many bursts which are closely spaced and nearly simultaneous.

Morton et al.[1] obtained a closed-form solution for a uniformly and stably stratified fluid. More general one-dimensional finite-difference cloud-rise codes which utilize the same basic assumptions, have been used in nuclear cloud-rise calculations. The basic assumptions are: 1) an incompressible fluid, except that small differences in (potential) density are allowed in buoyancy terms (Boussinesq approximation), and 2) the Taylor[4] entrainment hypothesis. One example of such a code is the cloud-rise module[5] of the DOD Land Fallout Prediction System—DELFIC. The density differences due to a nuclear burst are not initially small. This problem is avoided by choosing the initial time for cloud rise to be after the pressure in the fireball has returned to ambient. Existing nuclear burst data have been used[6] to determine initial conditions as a function of weapon yield and height of burst.

Analysis

Consider an initial cloud in the form of a circular disk of diameter $2b_0$ and thickness $\Delta h_0 = 2b_0/a$. The following assumptions are made:

1) The fluid is incompressible, except that small $[(\Delta\rho/\rho) \ll 1]$ variations in density are allowed in buoyancy terms.

2) Profiles of velocity and buoyancy through the cloud have the same form.

3) The rate of entrainment of ambient air at the cloud boundary is proportional to the mean vertical velocity of the cloud.

4) The entraining surface is taken to be the two faces of the disk. The rim area is small and the analysis is simpler without it.

With these assumptions, conservation of mass, momentum and energy (sometimes referred to as conservation of density deficiency or buoyancy) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi b^3}{a}\right) = 2\pi b^2 \alpha u \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi b^3}{a}\rho u\right) = \left(\frac{2\pi b^3}{a}\right) g(\rho_0-\rho) \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2\pi b^3}{a}(\rho_I-\rho)\right] = 2\pi b^2 \alpha u(\rho_I-\rho_0) \tag{3}$$

and a fourth equation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \tag{4}$$

Table 1 Transformation equations

Point source	Extended source
$b=\left(\frac{3}{\pi}\right)^{1/4}\alpha^{1/4}F_0^{1/4}G^{-1/4}R$	$b=\left(\frac{a}{\pi}\right)^{1/4}\alpha^{1/4}F_0^{1/4}G^{-1/4}R$
$u=\frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{1/4}U$	$u=\frac{3}{4a}\left(\frac{a}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{1/4}U$
$g\frac{\rho_0-\rho}{\rho_1}=\frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{3/4}\Delta$	$g\frac{\rho_0-\rho}{\rho_1}=\frac{3}{4a}\left(\frac{a}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{3/4}\Delta$
$x=\frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{-1/4}X$	$x=\frac{3}{4a}\left(\frac{a}{\pi}\right)^{1/4}\alpha^{-3/4}F_0^{1/4}G^{-1/4}X$
$t=G^{-1/2}t_1$	$t=G^{-1/2}t_1$

These equations are identical with the corresponding equations for a spherical cloud,[1] except that the cloud volume and area appear as $2\pi b^3/a$ and $2\pi b^2$, respectively. A suitable stretching of coordinates can be found to make them exactly identical. The transformation is

$$\frac{x_{ES}}{x_{PS}}=\frac{u_{ES}}{u_{PS}}=\left[g\left(\frac{\rho_0-\rho}{\rho_1}\right)\right]_{ES}\Big/\left[g\left(\frac{\rho_0-\rho}{\rho_1}\right)\right]_{PS}=\left(\frac{3}{a}\right)^{3/4},$$

$$\frac{b_{ES}}{b_{PS}}=\left(\frac{a}{3}\right)^{1/4},\quad \frac{t_{ES}}{t_{PS}}=1 \tag{5}$$

Note that these ratios are unity for $a=3$. This is the value of a for which the surface-to-volume ratio for the disk, with the rim excluded, is the same as for the sphere.

The boundary conditions used in Ref. 1 for a spherical cloud (and appropriate here also) are:

$$x=b=b^3u=0 \tag{6a}$$

and

$$F=F_0 \text{ at } t=0 \tag{6b}$$

The quantity F_0 is $(1/\rho_1)\times$total bouyancy released from the source at $t=0$. The orgin at $t=0$ is a virtual origin. Any real cloud is at first substantially different from this simple model. The solution of equations corresponding to Eqs. (1-4), but for a spherical cloud with the boundary conditions (6) is given in Ref. 1 and reproduced here in Fig. 1. This solution, together with Eq. (5), is the desired result. Table 1 gives the transformations which relate the nondimensional variables of Fig. 1 and the corresponding dimensional variables for both the point source and extended source clouds.

Reference 1 reports experiments in a stably stratified salt solution with a known amount of lighter fluid suddenly released from a small reservoir at the bottom of the tank. The entrainment parameter α was found to be approximately 0.3.

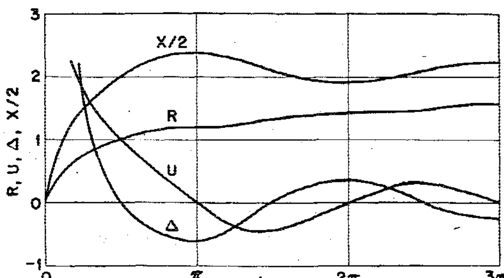

Fig. 1 Nondimensional cloud rise solution for both the point source and extended source.

A value of 0.2 is found to give reasonable agreement with nuclear cloud-rise data.

Results and Discussion

Consider the stabilization altitude $H_{ES}(N)$ of the cloud due to a field of N simultaneous explosions of equal yield.† Denote the entrainment constants for the single burst and multiburst clouds α_{SB} and α_{MB}, respectively, and let s be charge separation. Two cases are considered: 1) s=constant, 2) b=constant, with the dependence of H on N sought. In the first case, $F_0(N)=NF_0(1)$ and $\pi b_0^2=\beta N(\pi s^2/4)$ where β is a number slightly larger than unity because the sum of the areas of small circles which just touch is smaller than the area of the circle encompassing them. Then $a=2b_0/\Delta h_0=\sqrt{\beta N}(s/\Delta h_0)$ and from Table 1

$$\frac{H_{ES}(N)}{H_{PS}(1)}=\left\{\frac{3\Delta h_0\alpha_{SB}}{\sqrt{\beta}s\alpha_{MB}}\right\}^{3/4}N^{-1/8} \tag{7}$$

In the second case, the definition of $a(=2b_0/\Delta h_0)$ is used directly and

$$\frac{H_{ES}(N)}{H_{PS}(1)}=\left\{\frac{3\Delta h_0\alpha_{SB}}{2b_0\alpha_{MB}}\right\}^{3/4}N^{1/4} \tag{8}$$

Equation (7) is the more interesting result. First, note that $H_{ES}(1)>H_{PS}(1)$ for reasonable choices of s and Δh_0, e.g., $s=\Delta h_0$. This is a consequence of neglecting the rim area. The dependence on N should be predicted adequately by this simple model. Assuming that $\alpha_{SB}=\alpha_{MB}$, Eq. (7) indicates that the stabilization altitude slowly decreases with increasing N. This is as expected since the surface-to-volume ratio increases as N increases. Equation (8) is a restatement of the single burst result that the cloud height is proportional to the one-fourth power of the initial buoyancy or energy released.

The entrainment constant is probably lower for a multiburst cloud than for a single burst; this would tend to raise the stabilization altitude. Experiments[7] with thermals in a uniform environment yielded $\alpha=0.25$‡ about the same value as found in Ref. 1 in a stratified environment. But for a continuous source, α is about 0.1[2] for both point and line sources in a uniform environment. The main difference is that a plume from a continuous source does not have a bottom and

†A numerical example is worked out in Ref. 1 of a single explosion of 45.4 kg (100 lb) of TNT. The cloud rises to about 200 m in a still, standard atmosphere.

‡Recent Russian experiments[8] yielded $\alpha=0.18$-0.3, with the lowest value corresponding to the smallest disturbance in creating the thermal—done by blowing light gas into a soap bubble and pricking it with a needle. Both hydrogen and methane thermals rising in air were investigated.

the entrainment into thermals occurs primarily near the rear stagnation point. The path from the rim to the rear stagnation point of a multiburst thermal is long, and it seems likely that the entrainment would be less efficient, i.e., that α would be smaller.

It is hoped that the entrainment constant can be determined experimentally. It should not be difficult to do experiments similar to those that have been done before, but with many small sources rather than one source.

The consequences of the change in cloud geometry have been worked out using the closed-form solution of Ref. 1; but the same change in geometry could be made in a more general analysis and the same general conclusions would be expected.

Acknowlegment

This work was supported by the Space and Missile Systems Organization, Air Force Systems Command, under contract F04704-76-C-0013.

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Similar Solutions for Unsteady Transonic Flow

Wilson C. Chin*
Boeing Commercial Airplane Company, Seattle, Wash.

Introduction

THIS Note discusses the boundary-value problem for a class of similar solutions in unsteady transonic flow. The formulation introduced here furnishes test cases against which approximate time integration schemes may be checked. Extensions are also outlined for slender body flows.

Received Oct. 25, 1977; revision received Feb. 8, 1978.

Index categories: Nonsteady Aerodynamics; Subsonic Flow; Transonic Flow.

*Specialist Engineer, Aerodynamics Research Group. Member AIAA.

Analysis

Exact solutions in transonic flow are rare, and this is particularly true in three-dimensional unsteady problems. The usual idea in fluid mechanics is to reduce the number of dimensions by exploiting various physical symmetries. We shall consider slow temporal accelerations governed by

$$\frac{\partial}{\partial t}(-\Phi_x) + \frac{\partial}{\partial x}\left(K\Phi_x - \frac{1}{2}\Phi_x^2 - \Phi_t\right) + \frac{\partial}{\partial y}(\Phi_y) + \frac{\partial}{\partial z}(\Phi_z) = 0 \tag{1}$$

where K is the inviscid similarity parameter, Φ is the perturbation potential, and x, y, z and t are the usual independent variables. Separable solutions cannot be obtained in the classical sense because of the transonic nonlinearity. However, the Ansatz $\Phi(x,y,z,t) = A\varphi\ (\eta,\zeta,\xi)$ leads to simplifications if $\eta = x/A$, $\zeta = y/A$, $\xi = z/A$, and A is a function of time. This results in

$$\frac{(K-\varphi_\eta)\varphi_{\eta\eta} + \varphi_{\zeta\zeta} + \varphi_{\xi\xi}}{-2(\eta\varphi_{\eta\eta} + \zeta\varphi_{\eta\zeta} + \xi\varphi_{\eta\xi})} = \frac{\mathrm{d}A(t)}{\mathrm{d}t} \tag{2}$$

Now, in general, the left side is independent of time and the right side is independent of space. It follows that each must be equal to the same constant λ and, further, that $A = A_0 + \lambda t$. Equation (2) also defines the boundary-value problem for $\varphi(\eta,\zeta,\xi)$. For subsonic freestreams all disturbances decay far away. Since this must be true for every instant of time, regularity conditions must apply to φ. Neumann boundary conditions on the plane $z=0$ take the form $\Phi_z(x,y,0) = \varphi_\xi(\eta,\zeta,0)$. Thus the normal velocity is prescribed in terms of x/A and y/A.

One final question is the form of the required jump conditions. This is important in shock-fitting analyses. It is answered by recognizing Eq. (1) as the correct small-disturbance expression of mass continuity in proper conservation form.[1] To this we add $[\Phi] = 0$, as well as those jump conditions corresponding to $u_y - v_x = 0$, $u_z - w_x = 0$, and $w_y - v_z = 0$ (where $u,v,w = \Phi_{x,y,z}$), all rewritten,[2] of course, in terms of φ. For example, consider two space dimensions x and z. Let $x = f(z,t)$ describe the shock displacement. Then the shock slope (holding time fixed) is

$$\frac{\partial f}{\partial z} = -\frac{[\Phi_z]}{[\Phi_x]} = -\frac{[\varphi_\xi]}{[\varphi_\eta]}$$

while the shock speed (holding z fixed) is

$$\frac{\partial f}{\partial t} = -\frac{[K\Phi_x - \frac{1}{2}\Phi_x^2 - \Phi_t]}{[\Phi_x]} - \frac{[\Phi_z]^2}{[\Phi_x]^2} =$$

$$-\frac{[K\varphi_\eta - \frac{1}{2}\varphi_\eta^2 + \lambda(\eta\varphi_\eta + \xi\varphi_\xi - \varphi)]}{[\varphi_\eta]} - \frac{[\varphi_\xi]^2}{[\varphi_\eta]^2}$$

Formulas for the three-dimensional case can be written similarly without difficulty. The similarity analysis is easily generalized for slender body flows. The Ansatz used is $\Phi = A\varphi(\eta,\rho,\theta)$, where $\eta = x/A$, $\rho = r/A$ is the radial coordinate, and θ is the azimuthal variable. Putting $\Phi_{rr} + r^{-1}\Phi_r + r^{-2}\Phi_{\theta\theta}$ in Eq. (1) again leads to a separable system, the corresponding boundary conditions being similarly separable.

Discussion

The solutions just introduced describe a restricted class of unsteady flows but they are useful in some limited aerodynamic applications. Because time is eliminated explicitly in the boundary-value formulation, φ can be solved by a modified Murman-Cole algorithm. The results provide a useful source of test cases against which approximate time